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Almost rank three graphs

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In celebration of a man of vision and a man of action.

Abstract

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A finite graph Γ is 'pair-symmetric' if Aut Γ acts transitively on each of the following four sets of ordered pairs of vertices: (i) edges which lie in a triangle, (ii) edges which lie in no triangle, (iii) non-edges which lie in a cotriangle, (iv) non-edges which lie in no cotriangle. A pair-symmetric graph Γ is vertex-transitive, so the definition implies that Aut Γ has rank $\leqslant 5$ on vertices. It turns out that Aut Γ has rank $\leqslant 4$, and nearly always has rank 3. We show that, with the exception of one (possibly empty) family of graphs, either Aut Γ has rank $\leqslant 3$ on vertices, or Γ is obtained in a natural way from certain distance-transitive antipodal coverings of complete graphs.

In a rank three graph Γ all ordered edges are equivalent under Aut Γ, as are all ordered non-edges. From a graph-theoretic point of view this assumption may appear somewhat extreme. For example, in the triangular prism $\bar{C}_6$, an edge which lies in a triangle could not possibly be equivalent to an edge which does not lie in a triangle; nevertheless any two edges (resp. non-edges) which are 'combinatorially alike' are in fact equivalent under Aut $\bar{C}_6$. The following definition captures what we mean by 'combinatorially alike'.

Definition. An undirected, finite graph Γ is *pair-symmetric* if Aut Γ acts transitively on (i) ordered edges which belong to a triangle, (ii) ordered edges which belong to no triangle, (iii) ordered non-edges which belong to a cotriangle, and (iv) ordered non-edges which belong to no cotriangle.

Theorem 1. *The graph Γ is pair-symmetric if and only if*

(i) *Γ is distance-transitive and* Aut Γ *has rank $\leqslant 3$ on vertices, or*

(ii) *Γ or $\bar{\Gamma}$ is obtained from a distance-transitive graph Δ of diameter 3 by joining each pair of vertices at distance 3, where Δ may be*

(a) *any 2-fold antipodal covering of a complete graph,*

(b) *any r-fold antipodal covering ($r \geqslant 3$) of a complete graph having girth $\geqslant 4$,*

(c) *any non-bipartite graph with $a_1 = a_3 = 0$, $b_2 \geqslant 2$.*

The graphs in (ii)(a) correspond in a uniform way to doubly-transitive permutation groups [6]. Theorems 1 and 2 in [4] give restrictions on the existence of antipodal coverings of the kind occurring in part (ii)(b). No examples of type (ii)(c) appear to be known. Our notation will be standard: see [2] for information about distance-transitive graphs, and [3, 5] for information about antipodal covering graphs.

Proof. It is not hard to see that the graphs mentioned in the theorem are all pair-symmetric. We must prove that they are the only such graphs.

Let Γ be pair-symmetric. Then $\bar{\Gamma}$ is also pair-symmetric. If u, v are vertices of Γ, then the ordered edge, or non-edge, (u, v) is of the same type as (v, u). Hence any pair of vertices of Γ can be interchanged. In particular, Γ is vertex-transitive. If Γ is disconnected, with t ($\geqslant 2$) components, then each component is a complete graph K_n (otherwise we would have ordered non-edges (u, v) and (u, w), with u, v in the same component and u, w in different components, and both these non-edges would lie in the cotriangle $\langle u, v, w\rangle$, but could not be equivalent under $\operatorname{Aut}\Gamma$). Observe that such graphs $t\cdot K_n$ and their complements, the regular complete multipartite graphs $K_{t;n}$, are distance-transitive with rank $\leqslant 3$, and so are included under part (i) of the theorem.

We may therefore assume that both Γ and its complement $\bar{\Gamma}$ are connected. The complete graphs K_n (and their complements $\bar{K}_n$) are also included under part (i) of the theorem. We may therefore further assume that Γ and $\bar{\Gamma}$ have diameter $\geqslant 2$. We consider first the case where Γ has diameter $\geqslant 3$.

Lemma 2. *Let Γ be a pair-symmetric graph. If Γ has diameter $\geqslant 3$, then Γ is a distance-transitive 2-fold antipodal covering of a complete graph. In particular,* $\operatorname{Aut}\Gamma$ *acts transitively on ordered edges and has two orbits on non-edges.*

Proof. Γ must have diameter exactly 3 (otherwise there would exist ordered non-edges (u, v) and (u, w) with $d(u, v) = 2$ and $d(u, w) = 4$, and both these non-edges would lie in the cotriangle $\langle u, v, w\rangle$ but could not be equivalent under $\operatorname{Aut}\Gamma$).

Let $\Gamma_i(u)$ denote the set of vertices of Γ at distance i from u. If $v \in \Gamma_2(u)$ and $w \in \Gamma_3(u)$, then (u, v) must lie in a cotriangle (otherwise

$$\Gamma_1(v) \supseteq \Gamma_3(u) \cup (\Gamma_2(u) - \{v\}) \cup (\Gamma_1(v) \cap \Gamma_1(u)),$$

whereas $\Gamma_1(w) \subseteq (\Gamma_3(u) - \{w\}) \cup \Gamma_2(u)$, so $|\Gamma_1(v)| > |\Gamma_1(w)|$). But then, since (u, v) and (u, w) are obviously not equivalent under $\operatorname{Aut}\Gamma$, (u, w) cannot lie in a cotriangle. Hence

$$\Gamma_1(w) = (\Gamma_3(u) - \{w\}) \cup \Gamma_2(u).$$

It follows that $\Gamma_1(u) = \Gamma_2(w)$, and that $\Gamma_3(w) = \{u\}$. Interchanging u and w we see that $\{w\} = \Gamma_3(u)$, $\Gamma_2(u) = \Gamma_1(w)$, so Γ is a 2-fold antipodal covering of a complete graph. Since $\mathrm{Aut}\,\Gamma$ has at most two orbits on ordered non-edges, the stabiliser $(\mathrm{Aut}\,\Gamma)_u$ must act transitively on $\Gamma_2(u)$. Similarly $(\mathrm{Aut}\,\Gamma)_w = (\mathrm{Aut}\,\Gamma)_u$ must act transitively on $\Gamma_2(w) = \Gamma_1(u)$. Thus Γ is distance-transitive. □

Thus if Γ or $\bar{\Gamma}$ has diameter $\geqslant 3$, they both occur under part (ii)(a) of the theorem. Thus we may assume that both Γ and $\bar{\Gamma}$ have diameter $\leqslant 2$. Taking complements if necessary, we may also assume that $|\Gamma_1(u)| \leqslant |\Gamma_2(u)|$ for each vertex u.

Lemma 3. *Let Γ be a connected, pair-symmetric graph for which both Γ and $\bar{\Gamma}$ have diameter 2. If $|\Gamma_1(u)| \leqslant |\Gamma_2(u)|$ for each vertex u, then either* (a) *$\Gamma = C_5$, or* (b) *every non-edge lies in a cotriangle. In each case $(\mathrm{Aut}\,\Gamma)_u$ acts transitively on $\Gamma_2(u)$. In particular, if $|\Gamma_1(u)| = |\Gamma_2(u)|$, then Γ is distance-transitive.*

Proof. Suppose we are not in case (b); i.e., some non-edge (u, v) lies in no cotriangle. Then

$$\Gamma_1(v) = (\Gamma_2(u) - \{v\}) \cup (\Gamma_1(v) \cap \Gamma_1(u)),$$

and $|\Gamma_1(v)| = |\Gamma_1(u)| \leqslant |\Gamma_2(u)|$, so $|\Gamma_1(u)| = |\Gamma_2(u)| = k$ (say). It also follows that $|\Gamma_1(v) \cap \Gamma_1(u)| = 1$ whenever the non-edge (u, v) lies in no cotriangle.

Consider first the case where *no* non-edge lies in a cotriangle. Then $|\Gamma_1(v) \cap \Gamma_1(u)| = 1$ for every $v \in \Gamma_2(u)$, so exactly k edges join $\Gamma_1(u)$ to $\Gamma_2(u)$. Hence $|\Gamma_1(w) \cap \Gamma_2(u)| = 1$ for each $w \in \Gamma_1(u)$ (otherwise we would have $\Gamma_1(w) \cap \Gamma_2(u) = \emptyset$ for some, but not all, $w \in \Gamma_1(u)$; but then $\Gamma_1(w) = \{u\} \cup (\Gamma_1(u) - \{w\})$, so choosing $w' \in \Gamma_1(u)$ with $\Gamma_1(w') \cap \Gamma_2(u) \neq \emptyset$ we would get ordered edges (u, w) and (u, w') which both lie in the triangle $\langle u, w, w' \rangle$, but which are not equivalent under $\mathrm{Aut}\,\Gamma$). Thus $\langle \Gamma_1(u) \rangle$ is regular of valency $k-2$, whereas if $v \in \Gamma_2(u)$ and $\{w\} = \Gamma_1(v) \cap \Gamma_1(u)$, the vertex w is an isolated vertex in $\langle \Gamma_1(v) \rangle$. Hence $k = 2$ and $\Gamma = C_5$.

If we let

$$\Gamma_{21}(u) = \{v \in \Gamma_2(u) \colon u, v \text{ lie in no } \bar{K}_3\},$$
$$\Gamma_{22}(u) = \{v \in \Gamma_2(u) \colon u, v \text{ lie in a } \bar{K}_3\},$$

then we have just shown that $\Gamma_{22}(u) = \emptyset$ implies $\Gamma = C_5$ and we are in case (a). If $\Gamma_{21}(u) = \emptyset$, then we are obviously in case (b).

Thus we may assume that $\Gamma_{21}(u) \neq \emptyset \neq \Gamma_{22}(u)$, whence $k \geqslant 3$. Γ has $2k+1$ vertices so the valency must be even, $k \geqslant 4$. Recall that $|\Gamma_1(v) \cap \Gamma_1(u)| = 1$ whenever $v \in \Gamma_{21}(u)$. Let

$$V(u) = \{w \in \Gamma_1(u) \colon \{w\} = \Gamma_1(v) \cap \Gamma_1(u) \text{ for some } v \in \Gamma_{21}(u)\}.$$

Then $V(u)$ is non-empty and is an $(\mathrm{Aut}\,\Gamma)_u$-orbit (since $\Gamma_{21}(u)$ is). Hence either

(i) $V(u) = \{w \in \Gamma_1(u) \colon u, w \text{ lie in no } K_3\}$, or

(ii) $V(u) = \{w \in \Gamma_1(u) \colon u, w \text{ lie in some } K_3\}$

Let $w \in V(u)$. Then $\{w\} = \Gamma_1(v) \cap \Gamma_1(u)$ for some $v \in \Gamma_{21}(u)$.

Suppose case (i) occurs. Then w is an isolated vertex in $\langle \Gamma_1(u) \rangle$, whence $|\Gamma_1(w) \cap \Gamma_2(u)| = k-1$. But $\Gamma_1(v) \supseteq \Gamma_2(u) - \{v\}$ (or (u, v) would lie in a cotriangle). Thus w has valency $k-2$ in $\Gamma_1(v)$; so some vertex in $\Gamma_1(u)$ has valency $k-2$ in $\Gamma_1(u)$. Hence $\langle \Gamma_1(u) \rangle = K_{k-1} \cup K_1$, $|V(u)| = |\Gamma_{21}(u)| = 1$. But then, since $k \geq 4$, the transitivity of $(\operatorname{Aut} \Gamma)_u$ on $\Gamma_{22}(u) = \Gamma_2(u) - \{v\}$ would mean that no vertex could be isolated in $\langle \Gamma_{22}(u) \rangle$, let alone in $\Gamma_1(v) \rangle = \langle \Gamma_{22}(u) \cup \{w\} \rangle$. Thus case (i) does not occur.

Suppose case (ii) occurs. Since $w \in V(u)$, u and w must lie in some triangle $\langle u, w, w' \rangle$ (say). Then $w' \in V(u)$, so $|V(u)| \geq 2$. Clearly $|V(u)| \leq |\Gamma_{21}(u)|$. And $|\Gamma_{21}(u)| < k$ (since $\Gamma_{22}(u) \neq \emptyset$). Thus we may choose $w'' \in \Gamma_1(u) - V(u)$. But then w'' is adjacent to $k-1$ vertices in $\Gamma_2(u)$, but to no vertex in $\Gamma_{21}(u)$. Hence $|\Gamma_{21}(u)| = 1$, so $|V(u)| = 1$, a contradiction. □

It will be convenient to reformulate Lemma 3 as follows.

Corollary 4. *Let Γ, $\bar{\Gamma}$ be pair-symmetric of diameter 2, and suppose $|\Gamma_1(u)| \leq |\Gamma_2(u)|$ for each vertex u of Γ. If* Aut Γ *does not act as a rank three group on Γ, then*

(i) $|\Gamma_1(u)| < |\Gamma_2(u)|$,

(ii) $(\operatorname{Aut} \Gamma)_u$ *acts transitively on $\Gamma_2(u) = \Gamma_{22}(u)$ (since very non-edge lies in a cotriangle), and*

(iii) $(\operatorname{Aut} \Gamma)_u$ *has two nontrivial orbits on $\Gamma_1(u)$, namely*

$$\Gamma_{11}(u) = \{w \in \Gamma_1(u) \colon u, w \text{ lie in a } K_3\}$$

and

$$\Gamma_{12}(u) = \{w \in \Gamma_1(u) \colon u, w \text{ lie in no } K_3\}.$$

To complete the proof of Theorem 1 we need only identify the graphs Γ of Corollary 4 (and their complements). Let Γ be such a graph, and consider the associated graph $\Delta = \Gamma_{12}$ with the same vertex set as Γ, but with u and v adjacent in Δ if and only if $v \in \Gamma_{12}(u)$. (Thus Δ is the graph obtained from Γ by erasing all those edges of Γ which lie in a triangle of Γ.) It is easy to see that the graph Δ is connected with diameter 3 and girth ≥ 4. Aut Γ = Aut Δ acts distance-transitively on Δ, so Δ is distance-regular of valency $|\Delta(u)| = m$ (say). Since no two vertices of $\Gamma_{11}(u) = \Delta_3(u)$ are joined in Δ, Δ has intersection array

$$\begin{pmatrix} * & 1 & c & m \\ 0 & 0 & a & 0 \\ m & m-1 & b & * \end{pmatrix}$$

where $c + a + b = m$. Now let $|\Gamma_{11}(u)| = r - 1 \geq 2$. Then $b = 1$ if and only if $\Delta = (r \cdot K_{m+1})_c$ is a distance-transitive r-fold antipodal covering of a complete graph for some $r \geq 3$, whence we are in case (ii)(b) of Theorem 1. If $b \geq 2$, the graph Γ (and $\bar{\Gamma}$) occurs under case (ii)(c) of Theorem 1. This completes the proof. □

One would like to either eliminate, or have examples of, the graphs lumped together under case (ii)(c). The corresponding graphs $\Delta = \Gamma_{12}$ (with $b \geq 2$) have several interesting properties. Firstly, if 'incidence' is interpreted as 'adjacency in Δ', then for each vertex u, the sets $\Delta_1(u)$ and $\Delta_2(u)$ are the points and blocks of a 2-design with $\lambda = c - 1$. Secondly, the graph Δ has a second distance-transitive structure in that the graph $\Sigma = \Delta_3 = \Gamma_{11}$ is also distance-transitive [1]. Thirdly, if 'incidence' is interpreted as 'adacency in Σ', then the sets $\Delta_1(u)$ and $\Delta_2(u)$ are the points and blocks of another 2-design with $\lambda = b(b-1)/c$. The existence of such graphs is one of the many open questions mentioned in [3].

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